

## IX.5 GENERALIZED FINITE INTEGRAL TRANSFORM METHOD



IX.5. Introduction	857
IX.5.1 Differential operator $L$	858
IX.5.2 Supplemental Sturm-Liouville Problem and vector space $L_p^2(x_1, x_2)$	858
IX.5.3 Finite Integral Transform pair	859
IX.5.4 Operational Property	860
IX.5.5 Outline of the Finite Integral Transform	862
IX.5.6 Example - Viscous fluid flow between two rotating cylinders	863
IX.5.7 Example – Heat Equation	864
IX.5.8 Example – Fokker-Planck Equation	867
IX.5.9. Example – Hanging Cable	869
IX.5.10. Example - Transient Heat Transfer in the Fin	868
IX.5.11. Example – The Rock	876
IX.5.14. Laminar flow in a Parallel-Plate Channel I. Prescribed Wall Temperature	887
IX.5.15. Laminar flow in a Parallel-Plate Channel II. Prescribed Wall Flux	901
IX.5.16. Laminar Slip-Flow in a Rectangular Duct	915
IX.5.16. Old Version of Introduction (Fall 2017)	931

*February 2019**June 2020**February 2021 (Lake Effect)**January 2022**December 2023*

### IX.5.1 THE GENERALIZED FINITE INTEGRAL TRANSFORM METHOD

#### Introduction

Let  $u(x, \xi)$  be a function of the independent variable  $x$  and of the some other independent variables denoted by the consolidated symbol  $\xi$ .

Consider a partial differential equation for  $u(x, \xi)$  in the domain for which the variable  $x$  is defined in the finite interval  $x_1 < x < x_2$ . Let  $L$  be a linear  $2^{nd}$  order differential operator applied to the function  $u$  with respect to the variable  $x$

$$Lu \equiv a_0(x)u'' + a_1(x)u' + a_2(x)u \quad x_1 < x < x_2, \quad \xi \in G_\xi \quad (1)$$

Assume that conditions are given for  $u(x, \xi)$  at the boundary of interval  $(x_1, x_2)$ :

$$x = x_1 \quad \left[ -k_1 \frac{\partial u}{\partial x} + h_1 u \right]_{x=x_1} = f_1(\xi) \quad (2)$$

$$x = x_2 \quad \left[ +k_2 \frac{\partial u}{\partial x} + h_2 u \right]_{x=x_2} = f_2(\xi) \quad (3)$$

#### Integral transform

$$\mathfrak{I}_x \{u(x, \xi)\} = \bar{u}(\xi)$$

Our objective is to construct the integral transform  $\mathfrak{I}_x$  in the variable  $x$

$$\mathfrak{I}_x \{u(x, \xi)\} \rightarrow \bar{u}(\xi)$$

which eliminates differential operator  $L$

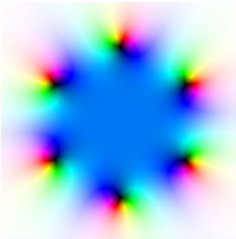
$$\mathfrak{I}_x \{Lu(x, \xi)\} = \text{some expression which includes } \bar{u}(\xi) \text{ but not } L.$$

In the Cartesian coordinate system for differential operator  $Lu \equiv \frac{\partial^2 u}{\partial x^2}$ , such an integral transform was defined by the integral kernel as the normalized eigenfunctions of the corresponding supplemental Sturm-Liouville Problem (see table SLP). Here, we consider an arbitrary differential operator  $L$ , but our approach also will be based on the solution of the corresponding eigenvalue problem.

#### Skip $\xi$ in derivation

For simplicity, the variable  $\xi$  will be omitted from the notations of  $u(x, \xi)$ .

Derivation will be concentrated only on the variable  $x$ . Therefore, notation  $u(x)$  does not eliminate the presence of the other variables or parameters.



The derived **generalized finite integral transform method** is termed generalized because it can be applicable for the wide family of the second order differential operators, and the traditional methods such as the Finite Fourier transform method and the Finite Hankel transform method will be its particular cases. Transform is called finite because the variable  $x$  belongs to the finite interval  $x_1 < x < x_2$ .

Construction of the finite integral transform  $\mathfrak{I}_x$  consists in the following steps.

### 1. Analyze Differential operator $L$

Consider the differential expression of the second order defined as a result of application of some operator  $L$  to the function  $u(x) \in C_2(x_1, x_2)$

$$Lu(x) \equiv a_0(x)u'' + a_1(x)u' + a_2(x)u, \quad x_1 < x < x_2 \quad (4)$$

where coefficients  $a_k(x) \in C[x_1, x_2]$  are continuous functions, and

$$a_0(x) > 0, \quad x_1 < x < x_2$$

Let the following boundary conditions are imposed on  $u(x)$ :

$$-\alpha_1 u'(x_1) + \beta_1 u(x_1) = f_1(\xi), \quad \alpha_1 > 0, \beta_1 > 0, \alpha_1 + \beta_1 > 0 \quad (5a)$$

$$+\alpha_2 u'(x_2) + \beta_2 u(x_2) = f_2(\xi), \quad \alpha_2 > 0, \beta_2 > 0, \alpha_2 + \beta_2 > 0 \quad (5b)$$

Define functions  $p(x), q(x)$ , and  $r(x)$  by the equations

$$p(x) = \frac{1}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)} dx} > 0 \quad p(x) > 0$$

$$q(x) = a_2(x)p(x) = \frac{a_2(x)}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)} dx}$$

$$r(x) = a_0(x)p(x) = e^{\int \frac{a_1(x)}{a_0(x)} dx} > 0 \quad r(x) > 0.$$

Then operator  $L$  can be rewritten in the following **self-adjoint form** (p.442)

**Self-adjoint form**

$$Lu(x) \equiv \frac{1}{p(x)} \left[ (ru')' + qu \right] \quad (6)$$

### 2. Supplemental eigenvalue problem and corresponding inner product vector space $L_p^2(x_1, x_2)$

Consider eigenvalue problem for the given operator  $L$  with a parameter  $\lambda$  subject to the boundary conditions, which are the homogeneous case of conditions (5):

**Eigenvalue problem**

$$Ly = \lambda y \quad x_1 < x < x_2 \quad (7)$$

$$-\alpha_1 y'(x_1) + \beta_1 y(x_1) = 0, \quad \alpha_1 > 0, \beta_1 > 0, \alpha_1 + \beta_1 > 0 \quad (8a)$$

$$+\alpha_2 y'(x_2) + \beta_2 y(x_2) = 0, \quad \alpha_2 > 0, \beta_2 > 0, \alpha_2 + \beta_2 > 0 \quad (8b)$$

Rewrite equation (7) using the self-adjoint form of operator (6):

$$\frac{1}{p} \left[ (ry')' + qy \right] = \lambda y$$

and rearrange this equation to the Sturm-Liouville form (p.438):

$$(ry')' + (q - \lambda p)y = 0 \quad (9)$$

According to Sturm-Liouville Theorem (VI.3, p.439), for this eigenvalue problem to be solvable, parameter  $\lambda$  has to be non-positive,  $\lambda = -\mu^2$ . Therefore,

$$(ry')' + (q + \mu^2 p)y = 0 \quad (9')$$

**Eigenvalues**  $\mu_n$

**Eigenfunctions**  $y_n(x)$

There exist infinitely many **eigenvalues**  $0 = \mu_0 < \mu_1 < \mu_2 < \dots$  ( $\mu_0 = 0$  only when both  $\alpha_1 = \alpha_2 = 0$ ), for which equation (9') has non-trivial solutions  $y_n(x)$

$$L y_n = -\mu_n^2 y_n \quad (10)$$

satisfying the homogeneous boundary conditions (8).

**Inner product**

Define inner product for  $v(x), u(x) \in L^2(x_1, x_2)$  with the weight function  $p(x)$  as

$$(v, w)_p = \int_{x_1}^{x_2} v(x) w(x) p(x) dx \quad (11)$$

Define the norm of functions  $v(x) \in L^2(x_1, x_2)$  with the help of inner product

**Norm**

$$\|v(x)\|_p^2 = (v, v)_p = \int_{x_1}^{x_2} v^2(x) p(x) dx \quad (12)$$

**Vector Space**

Then the vector space  $L_p^2(x_1, x_2)$  with the defined inner product is the Hilbert space (Riesz-Fischer Theorem, VI.1, p.432).

**Orthogonality of  $\{y_n(x)\}$**

According to Sturm-Liouville Theorem, the set of eigenfunctions  $\{y_n(x)\}$  is a complete set of mutually **orthogonal functions** in  $L_p^2(x_1, x_2)$

$$(y_n, y_m)_p = 0 \quad \text{if} \quad n \neq m \quad (13)$$

and any function  $f(x) \in L_p^2(x_1, x_2)$  can be represented by the series expansion as

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad \text{where} \quad c_n = \frac{(f, y_n)_p}{(y_n, y_n)_p} = \frac{(f, y_n)_p}{\|y_n\|_p^2} \quad (15)$$

Therefore, expansion of  $f(x)$  can be rewritten as

$$f(x) = \sum_{n=1}^{\infty} (f, y_n)_p \frac{y_n(x)}{\|y_n\|_p^2} \quad (16)$$

This eigenfunction expansion formula can be used for definition of the Finite Integral Transform, and for the corresponding Inverse Transform for reconstruction of the function from its transform:

### 3. Finite Integral Transform pair

**Direct Transform**

$$\mathfrak{T}\{u(x)\} = (u, y_n)_p = \int_{x_1}^{x_2} u(x) y_n(x) p(x) dx = \bar{u}_n \quad (17)$$

**Inverse Transform**

$$\mathfrak{T}^{-1}\{\bar{u}_n\} = \sum_{n=1}^{\infty} \bar{u}_n \frac{y_n(x)}{\|y_n\|_p^2} = u(x) \quad (18)$$

In a result, function  $u(x)$  of continuous variable  $x$  is transformed to a sequence of discrete values  $\bar{u}_n$ . Inverse transformation is defined by the series expansion.

#### 4. Operational Property

For application of this integral transform to solution of the boundary value problems, the operational property of integral transform has to be developed.

Let  $u(x)$  satisfies non-homogeneous boundary conditions (5), and let  $Lu$  be the differential operator defined by equations (4), which has the self-adjoint representation (6). Apply integral transform (17) to the differential operator  $L$  :

$$\begin{aligned}
 \mathfrak{I}\{Lu\} &= \mathfrak{I}\left\{\frac{1}{p(x)}\left[(ru')' + qu\right]\right\} = \int_{x_1}^{x_2} \frac{1}{\cancel{p(x)}} \left[(ru')' + qu\right] y_n(x) \cancel{p(x)} dx \\
 &= \int_{x_1}^{x_2} (ru')' y_n(x) dx + \int_{x_1}^{x_2} qu y_n(x) dx \\
 &= \int_{x_1}^{x_2} y_n(x) d(ru') + \int_{x_1}^{x_2} qu y_n(x) dx \\
 &= \left[y_n(x) ru'\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} y_n'(x) ru' dx + \int_{x_1}^{x_2} qu y_n(x) dx \\
 &= \left[y_n(x) ru'\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} y_n'(x) r du + \int_{x_1}^{x_2} qu y_n(x) dx \\
 &= \left[y_n(x) ru'\right]_{x_1}^{x_2} - \left[y_n'(x) ru\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} u \left[ry_n'(x)\right]' dx + \int_{x_1}^{x_2} qu y_n(x) dx \\
 &\Rightarrow \left[y_n(x) ru'\right]_{x_1}^{x_2} - \left[y_n'(x) ru\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} u \left[-qy_n - \mu_n^2 y_n p\right] dx + \int_{x_1}^{x_2} qu y_n(x) dx \\
 &= \left[y_n(x) ru'\right]_{x_1}^{x_2} - \left[y_n'(x) ru\right]_{x_1}^{x_2} - \mu_n^2 \int_{x_1}^{x_2} u y_n p dx - \int_{x_1}^{x_2} u q y_n dx + \int_{x_1}^{x_2} qu y_n(x) dx \\
 &= \left[y_n(x) ru'\right]_{x_1}^{x_2} - \left[y_n'(x) ru\right]_{x_1}^{x_2} - \mu_n^2 \bar{u}_n
 \end{aligned}$$

from Eqn.(9')

$$(ry_n')' = -qy_n - \mu_n^2 y_n p$$

$\Rightarrow$

$$\mathfrak{I}\{Lu\} = y_n(x_2)r(x_2)u'(x_2) - y_n(x_1)r(x_1)u'(x_1) - y_n'(x_2)r(x_2)u(x_2) + y_n'(x_1)r(x_1)u(x_1) - \mu_n^2 \bar{u}_n \quad (19)$$

This expression (19) can be used for derivation of the particular cases of the operational property for different types of boundary conditions. Consider the cases

##### I-I Dirichlet - Dirichlet

$$u(x_1) = f_1(\xi) \quad (5a)$$

$$u(x_2) = f_2(\xi) \quad (5b)$$

$$y(x_1) = 0 \quad (8a)$$

$$y(x_2) = 0 \quad (8b)$$

$$\begin{aligned}
 \mathfrak{I}\{Lu\} &= y_n(x_2)r(x_2)u'(x_2) - y_n(x_1)r(x_1)u'(x_1) - y_n'(x_2)r(x_2)u(x_2) + y_n'(x_1)r(x_1)u(x_1) - \mu_n^2 \bar{u}_n \\
 &= \cancel{y_n(x_2)r(x_2)u'(x_2)} - \cancel{y_n(x_1)r(x_1)u'(x_1)} - y_n'(x_2)r(x_2)u(x_2) + y_n'(x_1)r(x_1)u(x_1) - \mu_n^2 \bar{u}_n \\
 &= -\mu_n^2 \bar{u}_n - y_n'(x_2)r(x_2)u(x_2) + y_n'(x_1)r(x_1)u(x_1)
 \end{aligned}$$

$$\mathfrak{I}\{Lu\} = -\mu_n^2 \bar{u}_n - y_n'(x_2)r(x_2)f_2(\xi) + y_n'(x_1)r(x_1)f_1(\xi) \quad (20)$$

**I – II Dirichlet-Neumann**

$$u(x_l) = f_l(\xi) \quad (5a)$$

$$u'(x_2) = f_2(\xi) \quad (5b)$$

$$y(x_l) = 0 \quad (8a)$$

$$y'(x_2) = 0 \quad (8b)$$

$$\begin{aligned} \Im\{Lu\} &= y_n(x_2)r(x_2)u'(x_2) - \cancel{y_n(x_l)r(x_l)u'(x_l)} - \cancel{y'_n(x_2)r(x_2)u(x_2)} + y'_n(x_l)r(x_l)u(x_l) - \mu_n^2 \bar{u}_n \\ &= -\mu_n^2 \bar{u}_n + y_n(x_2)r(x_2)u'(x_2) + y'_n(x_l)r(x_l)u(x_l) \end{aligned}$$

$$\Im\{Lu\} = -\mu_n^2 \bar{u}_n + y_n(x_2)r(x_2)f_2(\xi) + y'_n(x_l)r(x_l)f_l(\xi) \quad (21)$$

**III-III Robin-Robin**

Boundary conditions are defined by equations (5) and (8).

Equations (8) can be rewritten as

$$y'(x_l) = +\frac{\beta_l}{\alpha_l} y(x_l), \quad \alpha_l > 0, \beta_l > 0$$

$$y'(x_2) = -\frac{\beta_2}{\alpha_2} y(x_2), \quad \alpha_2 > 0, \beta_2 > 0$$

$$\begin{aligned} \Im\{Lu\} &= y_n(x_2)r(x_2)u'(x_2) - y_n(x_l)r(x_l)u'(x_l) + \frac{\beta_2}{\alpha_2} y(x_2)r(x_2)u(x_2) + \frac{\beta_l}{\alpha_l} y(x_l)r(x_l)u(x_l) - \mu_n^2 \bar{u}_n \\ &= y_n(x_2)r(x_2) \left[ u'(x_2) + \frac{\beta_2}{\alpha_2} u(x_2) \right] + y_n(x_l)r(x_l) \left[ -u'(x_l) + \frac{\beta_l}{\alpha_l} u(x_l) \right] - \mu_n^2 \bar{u}_n \\ &= \frac{f_2}{\alpha_2} y_n(x_2)r(x_2) \left[ \alpha_2 u'(x_2) + \beta_2 u(x_2) \right] + \frac{f_l}{\alpha_l} y_n(x_l)r(x_l) \left[ -\alpha_l u'(x_l) + \beta_l u(x_l) \right] - \mu_n^2 \bar{u}_n \\ &= \frac{f_2}{\alpha_2} y_n(x_2)r(x_2) + \frac{f_l}{\alpha_l} y_n(x_l)r(x_l) - \mu_n^2 \bar{u}_n \end{aligned}$$

$$\Im\{Lu\} = -\mu_n^2 \bar{u}_n + \frac{f_2}{\alpha_2} y_n(x_2)r(x_2) + \frac{f_l}{\alpha_l} y_n(x_l)r(x_l) \quad (22)$$

**III – I Robin-Dirichlet**

$$-k_l u'(x_l) + h_l u(x_l) = f_l(\xi)$$

$$u(x_2) = f_2(\xi)$$

$$-k_l y'(x_l) + h_l y(x_l) = 0$$

$$y(x_2) = 0$$

$$\Im\{Lu\} = -\mu_n^2 \bar{u}_n + \quad (23)$$

## 5. OUTLINE of the Finite Integral Transform method

### 1) Given operator $L$

$$Lu(x) \equiv a_0(x)u'' + a_1(x)u' + a_2(x)u, \quad x_1 < x < x_2$$

where  $a_k(x) \in C[x_1, x_2]$ ,  $a_0(x) > 0$ ,  $x_1 < x < x_2$

Boundary conditions are imposed on function  $u(x)$

$$-\alpha_1 u'(x_1) + \beta_1 u(x_1) = f_1(\xi), \quad \alpha_1 \geq 0, \beta_1 \geq 0, \quad \alpha_1 + \beta_1 > 0$$

$$+\alpha_2 u'(x_2) + \beta_2 u(x_2) = f_2(\xi), \quad \alpha_2 \geq 0, \beta_2 \geq 0, \quad \alpha_2 + \beta_2 > 0$$

### Self-adjoint form

$$Lu(x) \equiv \frac{I}{p(x)} \left[ (ru')' + qu \right]$$

$$\text{where } p(x) = \frac{I}{a_0(x)} e^{\int \frac{a_1(x)}{a_0(x)} dx}, \quad q(x) = a_2(x)p(x), \quad r(x) = a_0(x)p(x)$$

### 2) Supplemental eigenvalue problem (Sturm-Liouville problem) $Ly = \lambda y$ , where $\lambda = -\mu^2$

$$(ry')' + (q + \mu^2 p)y = 0, \quad x_1 < x < x_2$$

$$-\alpha_1 y'(x_1) + \beta_1 y(x_1) = 0, \quad \alpha_1 \geq 0, \beta_1 \geq 0, \quad \alpha_1 + \beta_1 > 0$$

$$+\alpha_2 y'(x_2) + \beta_2 y(x_2) = 0, \quad \alpha_2 \geq 0, \beta_2 \geq 0, \quad \alpha_2 + \beta_2 > 0$$

Solve for **eigenvalues**  $\mu_n$ ,  $n = 0, 1, 2, \dots$  ( $\mu_0 = 0$  only if  $\alpha_1 = \alpha_2 = 0$ )

$$0 = \mu_0 < \mu_1 < \mu_2 < \dots$$

and for the corresponding **eigenfunctions**  $y_n(x)$

$$Ly_n = -\mu_n^2 y_n$$

satisfying the boundary conditions.

Hilbert space  $L_p^2(x_1, x_2)$

Define **inner product** for  $v, u \in L^2(x_1, x_2)$  with the weight function  $p(x)$  as

$$(v, w)_p = \int_{x_1}^{x_2} v(x)w(x)p(x)dx$$

and define the **norm** with the help of introduced weighted inner product

$$\|v\|_p^2 = (v, v)_p = \int_{x_1}^{x_2} v^2(x)p(x)dx$$

### 3) Integral Transform

$$\mathfrak{T}\{u(x)\} = (u, y_n)_p = \int_{x_1}^{x_2} u(x)y_n(x)p(x)dx = \bar{u}_n$$

### Inverse Transform

$$\mathfrak{T}^{-1}\{\bar{u}_n\} = \sum_{n=1}^{\infty} \bar{u}_n \frac{y_n(x)}{\|y_n\|_p^2} = u(x)$$

### 4) Operational property

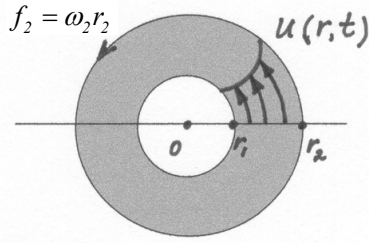
(the case of Robin-Robin b.c.'s)

$$\mathfrak{T}\{Lu\} = -\mu_n^2 \bar{u}_n + \frac{f_2}{\alpha_2} y_n(x_2)r(x_2) + \frac{f_1}{\alpha_1} y_n(x_1)r(x_1) \quad (\text{III-III})$$

5) Apply defined integral transform  $\mathfrak{T}\{\}$  to the differential equation using the operational property.

6) Solve for transformed function  $\bar{u}_n$  and apply the inverse transform to find the solution  $u(x) = \mathfrak{T}^{-1}\{\bar{u}_n\}$ .



**6. EXAMPLE Viscous fluid flow between two rotating cylinders** (see IX.4.7.5), *SF-AD-2-I-FiT example.mws*

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) - \frac{\nu}{r^2} u = \frac{1}{\alpha} \frac{\partial u}{\partial t}, \quad r_1 < r < r_2, \quad t > 0$$

$$\nu = l$$

initial condition:  $u(r, 0) = 0, \quad r_1 \leq r \leq r_2$

boundary conditions:  $\frac{\partial u(r_1)}{\partial r} = f_1(t) = c$  Neumann

$u_2(r_2) = f_2(t) = \omega_2 r_2$  Dirichlet

1) Differential operator  $L$

$$Lu(r) \equiv \frac{1}{r} \left[ (ru')' - \frac{1}{r} u \right] \quad \text{self-adjoint form, weight } p(r) = r$$

2) Sturm-Liouville problem

$$\frac{1}{r} \left[ (ry')' - \frac{1}{r} y \right] = -\mu^2 y \quad y'(r_1) = 0 \quad (\text{N})$$

$$y(r_2) = 0 \quad (\text{D})$$

$r^2 y'' + ry' + (\mu^2 r^2 - l)y = 0$  The Bessel equation of order  $\nu = l$   
Eigenvalues  $\mu_n$  are the positive roots of characteristic equation (p.518).

The corresponding solutions  $y_n(r)$  are (p.519):

$$y_n(r) = J_l(\mu_n r) / J_l(\mu_n r_2) - Y_l(\mu_n r) / Y_l(\mu_n r_2)$$

$$\|y_n(r)\|_r^2 = \int_{r_1}^{r_2} y_n^2(r) r dr \quad \text{squared norm} \quad (\text{see Ozisik, p.140})$$

3) Integral Transform

$$\mathfrak{I}\{u(x)\} = \bar{u}_n = (u, y_n)_{p=r} = \int_{r_1}^{r_2} u(r) y_n(r) r dr$$

4) Operational property

$$\mathfrak{I}\{Lu\} = -\mu_n^2 \bar{u}_n - r_1 f_1 y_n(r_1) - r_2 f_2 y_n'(r_2) \quad (\text{N-D, p.831 } \odot \odot)$$

5) Apply transform  $\mathfrak{I}$  to equation

$$\frac{\partial \bar{u}_n}{\partial t} = -\alpha \mu_n^2 \bar{u}_n - \alpha r_1 f_1 y_n(r_1) - \alpha r_2 f_2 y_n'(r_2)$$

Apply Laplace transform

$$s \bar{u}_n = -\alpha \mu_n^2 \bar{u}_n - \alpha [r_1 f_1 y_n(r_1) + r_2 f_2 y_n'(r_2)] \frac{1}{s}$$

6) Find the transformed solution  $\bar{u}_n$

$$\bar{u}_n = -\frac{1}{\mu_n^2} [r_1 f_1 y_n(r_1) + r_2 f_2 y_n'(r_2)] \left( \frac{1}{s} - \frac{1}{s + \alpha \mu_n^2} \right)$$

Apply inverse Laplace transform

$$\bar{u}_n = -\frac{1}{\mu_n^2} [r_1 f_1 y_n(r_1) + r_2 f_2 y_n'(r_2)] (1 - e^{-\alpha \mu_n^2 t})$$

Solution (apply inverse  $\mathfrak{I}^{-1}$ )

$$u(r, t) = \sum_{n=1}^{\infty} \bar{u}_n \frac{y_n(r)}{\|y_n\|_p^2} = -\sum_{n=1}^{\infty} \frac{r_1 f_1 y_n(r_1) + r_2 f_2 y_n'(r_2)}{\mu_n^2} \cdot \frac{y_n(r)}{\|y_n\|_p^2} (1 - e^{-\alpha \mu_n^2 t})$$

Steady state solution

$$Lu_s(r) = 0 \Rightarrow u_s(r) = \frac{(f_1 r_1^2 + f_2 r_2)}{r_1^2 + r_2^2} r + \frac{(f_2 - f_1 r_2) r_2 r_1^2}{r_1^2 + r_2^2} \frac{1}{r}$$

Not-oscillating form of solution

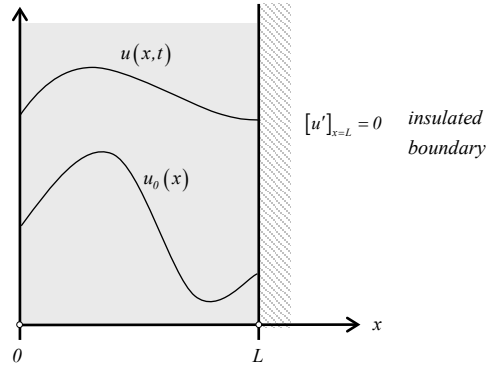
$$u(r, t) = u_s(r) + \sum_{n=1}^{\infty} \frac{r_1 f_1 y_n(r_1) + r_2 f_2 y_n'(r_2)}{\mu_n^2} \frac{y_n(r)}{\|y_n\|_p^2} e^{-\alpha \mu_n^2 t}$$



**7. EXAMPLE Heat Equation** (see IX.3.2)**Heat conduction in the 1-dimensional slab with heat generation**

left boundary is exposed  
to a convective environment  
with time dependent  
temperature of surroundings

$$\left[ -k_l \frac{\partial u}{\partial x} + h_l u \right]_{x=0} = \frac{h_l u_{l,\infty}(t)}{f_0(t)}$$



Equation  $\frac{1}{\alpha} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + S(x,t), \quad 0 < x < L, \quad t > 0$

Initial condition  $u(x,0) = u_0(x) \quad 0 \leq x \leq L$

Boundary conditions  $\left[ -k_l \frac{\partial u}{\partial x} + h_l u \right]_{x=0} = h_l u_{l,\infty}(t) = f_0(t) \quad \text{Robin}$

$$\left[ \frac{\partial u}{\partial x} \right]_{x=L} = 0 = f_L(t) \quad \text{Neumann}$$

1) Differential operator  $Lu \equiv \frac{\partial^2 u}{\partial x^2}$ , is already in self-adjoint form,  $p(x) = 1$ ,  $q(x) = 0$ ,  $r(x) = 1$ ,

2) Sturm-Liouville problem  $X'' = -\mu^2 X \quad \left[ -X' + H_l X \right]_{x=0} = 0 \quad \text{Robin} \quad H_l = \frac{h_l}{k_l}$

$$\left[ X' \right]_{x=L} = 0 \quad \text{Neumann}$$

solution (VI.5, p.448)  $X_n(x) = \cos \mu_n(x-L), \quad \|X_n(x)\|^2 = \frac{L}{2} + \frac{\sin(2\mu_n L)}{4\mu_n}$ ,

where  $\mu_n$  are the positive roots of  $\mu \sin \mu L - H_l \cos \mu L = 0$

3) Integral Transform  $\bar{u}_n(t) = \int_0^L u(x,t) X_n(x) dx = \int_0^L u(x,t) \cos[\mu_n(x-L)] dx$

Inverse Transform  $u(x,t) = \sum \bar{u}_n \frac{X_n(x)}{\|X_n\|^2}$

4) Operational property  $\mathfrak{I}\{Lu\} = -\mu_n^2 \bar{u}_n + y_n(x_2)r(x_2)f_2(\xi) + y'_n(x_1)r(x_1)f_1(\xi)$

$$= -\mu_n^2 \bar{u}_n + X'_n(0)h_l u_{l,\infty}(t)$$

$$= -\mu_n^2 \bar{u}_n - \mu_n \sin \mu_n(x-L) h_l u_{l,\infty}(t)$$

5) Transformed Equation  $\frac{1}{\alpha} \frac{\partial}{\partial t} \bar{u}_n(t) = -\mu_n^2 \bar{u}_n - \mu_n h_l \sin[\mu_n(x-L)] u_{l,\infty}(t) + \bar{S}_n(t)$

where  $\bar{S}_n(t) = \int_0^L S(x,t) X_n(x) dx$  is the transformed source function.

Transformed initial condition  $\bar{u}_n(0) = \int_0^L u_0(x) X_n(x) dx = \bar{u}_{0,n}$

6) Apply Laplace transform:

$$\frac{s}{\alpha} \hat{U}_n(s) - \frac{1}{\alpha} \bar{u}_{0,n} = \frac{1}{k_1} \hat{f}_0(s) X_n(0) - \mu_n^2 \hat{U}_n(s) + \hat{S}_n(s), \text{ where } \hat{U}_n(s) = \{\bar{u}_n(t)\}.$$

7) Solve for transformed function

$$\hat{U}_n(s) = \bar{u}_{0,n} \frac{1}{s + \alpha \mu_n^2} + \frac{\alpha}{k_1} X_n(0) \hat{f}_0(s) \frac{1}{s + \alpha \mu_n^2} + \alpha \hat{S}_n(s) \frac{1}{s + \alpha \mu_n^2}. \quad \text{Note that } \frac{1}{s + \alpha \mu_n^2} = L\{e^{-\alpha \mu_n^2 t}\}.$$

Apply inverse Laplace transform using convolution theorem:

$$\bar{u}_n(t) = L\{\hat{U}_n(s)\} = \bar{u}_{0,n} e^{-\alpha \mu_n^2 t} + \frac{\alpha}{k_1} X_n(0) [f_0(t) * e^{-\alpha \mu_n^2 t}] + \alpha [\bar{S}_n(t) * e^{-\alpha \mu_n^2 t}]$$

Which using convolution integral can be written as

$$\bar{u}_n(t) = L^{-1}\{\hat{U}_n(s)\} = \bar{u}_{0,n} e^{-\alpha \mu_n^2 t} + \frac{\alpha}{k_1} X_n(0) \left[ \int_{\tau=0}^t f_0(\tau) e^{-\alpha \mu_n^2 (t-\tau)} d\tau \right] + \alpha \left[ \int_{\tau=0}^t \bar{S}_n(\tau) e^{-\alpha \mu_n^2 (t-\tau)} d\tau \right]$$

8) Then the solution of the given initial-boundary value problem is found by the inverse transform:

$$u(x,t) = \sum_{n=1}^{\infty} \bar{u}_n \frac{X_n(x)}{\|X_n(x)\|^2}$$

**Consider the particular case of periodic surrounding temperature and zero source function and initial condition**

$$f_0(t) = h_1 f_0 \sin \omega t, \quad S(x,t) = 0, \quad u_0(x) = 0$$

Then transformed solution can be written as

$$\bar{u}_n(t) = \frac{\alpha}{k_1} X_n(0) \left[ \int_{\tau=0}^t f_0 h_1 \sin(\omega \tau) e^{-\alpha \mu_n^2 (t-\tau)} d\tau \right]$$

And after evaluation of convolution integral the transformed function becomes

$$\bar{u}_n(t) = \frac{\alpha}{k_1} f_0 h_1 X_n(0) \frac{\alpha \mu_n^2 \sin(\omega t) - \omega \cos(\omega t) + \omega e^{-\alpha \mu_n^2 t}}{\omega^2 + \alpha^2 \mu_n^4}$$

which reaches the periodic quasi-steady state

$$\bar{u}_n(t) = \frac{\alpha}{k_1} f_0 h_1 X_n(0) \frac{\alpha \mu_n^2 \sin(\omega t) - \omega \cos(\omega t)}{\omega^2 + \alpha^2 \mu_n^4}$$

FIT-01 RN.mws

Maple solution with  $L = 3, k_1 = 2, h_1 = 4, \alpha = 1, f_0(t) = f_0 \sin \omega t, f_0 = 1, \omega = 2$

